The Root Locus Method

MEM 355 Performance Enhancement of Dynamical Systems

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Outline

The root locus method was introduced by Evans in the 1950's. It remains a popular tool for simple SISO control design.

- What is a root locus?
- Poles & Transient Response (Why do we care about poles?)
- The Root Locus Method
  - Problem Definition
  - The Two Key Formulas
  - Root Locus Rules
- Examples:
  - Flexible Spacecraft
  - Robotic Arm
  - Helicopter Pitch Control

Designing a Feedback Control System Using The Root Locus

- First, we choose a compensator
  - There are many useful compensator types. We have already seen Proportional and Proportional plus Integral.
  - This gives us a control structure, i.e., a compensator transfer function.
- The compensator will have one or more free parameters.
  - The root locus method typically focuses on the gain parameter. It is an approach to select the gain as to achieve desired transient behavior.
  - The root locus rules of behavior provide insight for adjusting additional compensator parameters.
  - The root locus structure also yields ideas for adding elements to the compensator.

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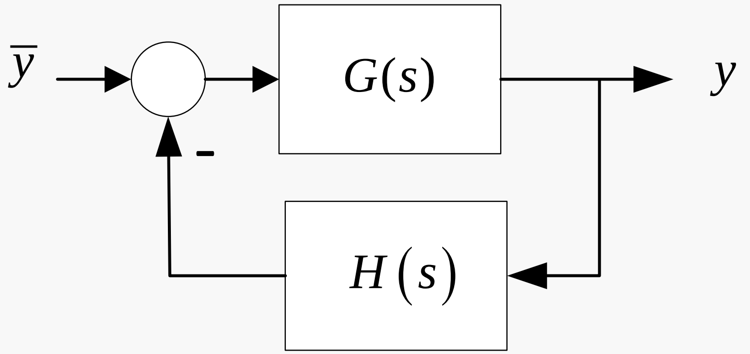
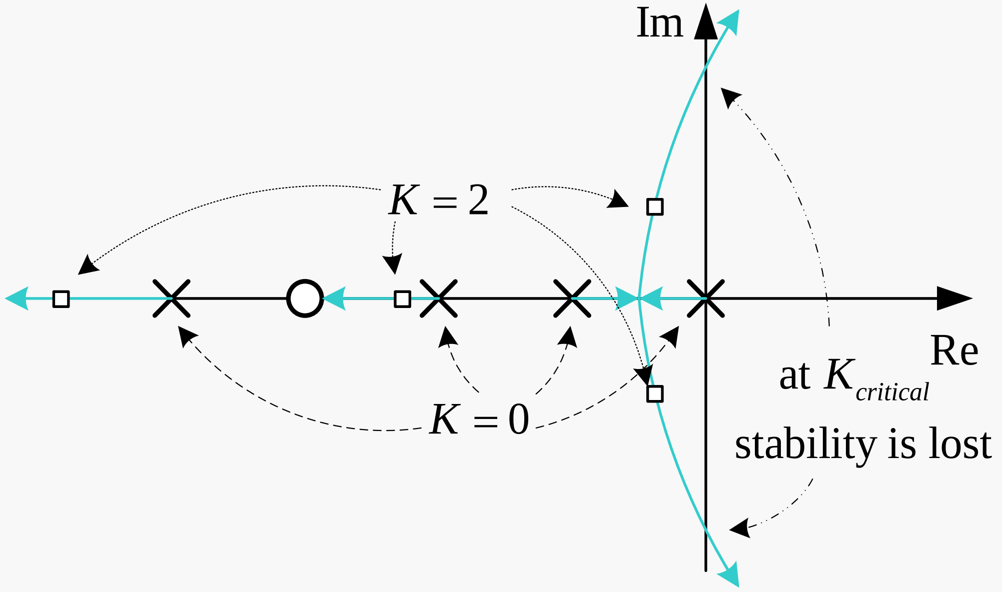
What is a Root Locus?

- On the right is a negative feedback loop
- We wish to examine the closed loop poles as the gain *K* varies
- As *K* increases from zero the 4 poles move from the open loop values & trace 4 loci
- At any particular value of *K* there are 4 closed loop poles
- In this example there is a critical value of *K* at which the system becomes unstable.



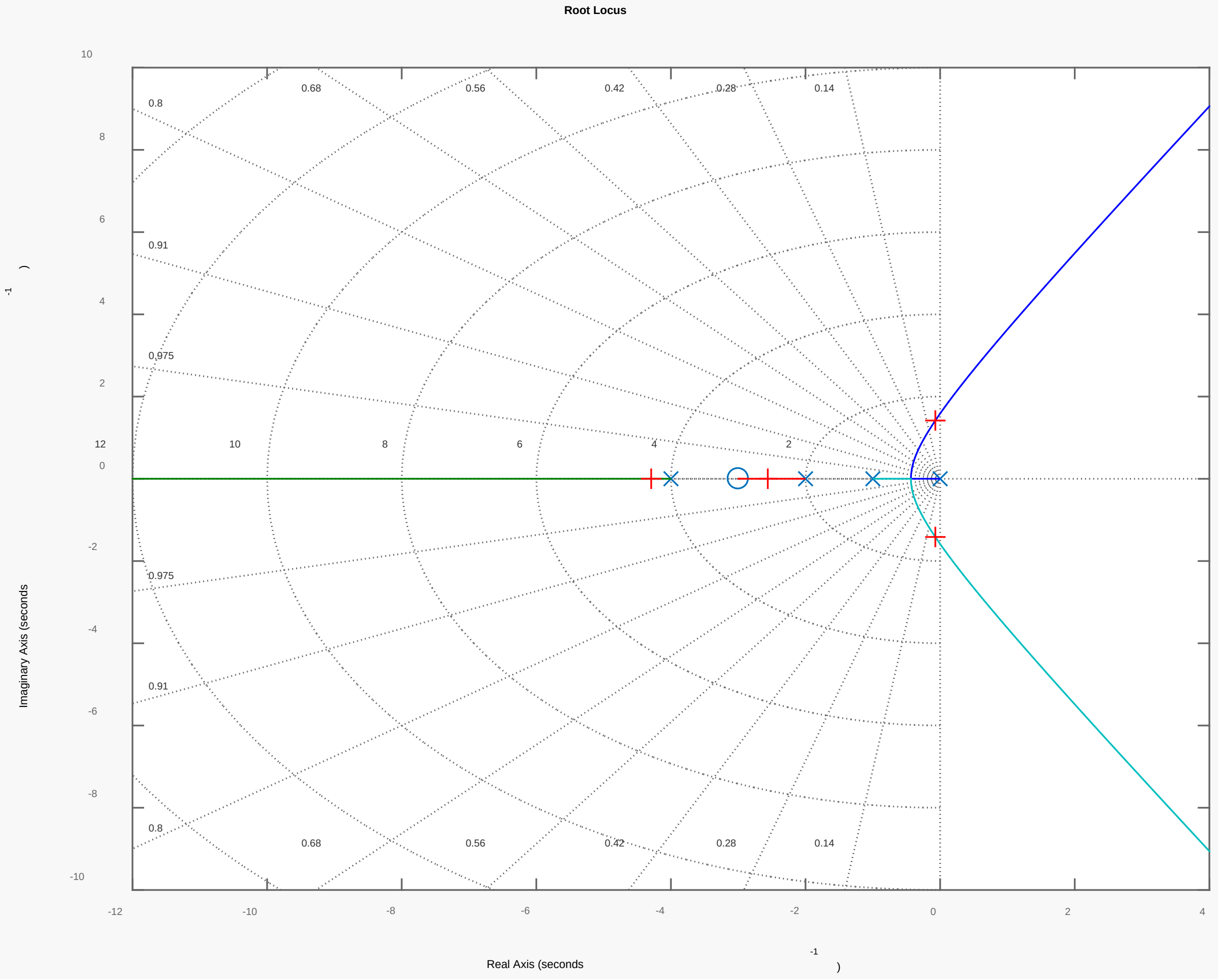
$$G(s)H(s) = K\frac{(s+3)}{s(s+1)(s+2)(s+4)}$$

$$G_{\bar{y}e} = \frac{1}{1+GH} = \frac{s(s+1)(s+2)(s+4)}{s^4+7s^3+14s^2+(8+K)s+3K}$$



Using Matlab

```
>> s=tf('s');
>> G=(s+3)/(s*(s+1)*(s+2)*(s+4));
>> rlocus(G)
>> sgrid
>> [K,Poles]=rlocfind(G)
Select a point in the graphics
window
selected_point =
  -0.0720 + 1.4161i
K =
    7.3729
Poles =
  -4.2940 + 0.0000i
  -2.5617 + 0.0000i
  -0.0722 + 1.4162i
  -0.0722 - 1.4162i
```

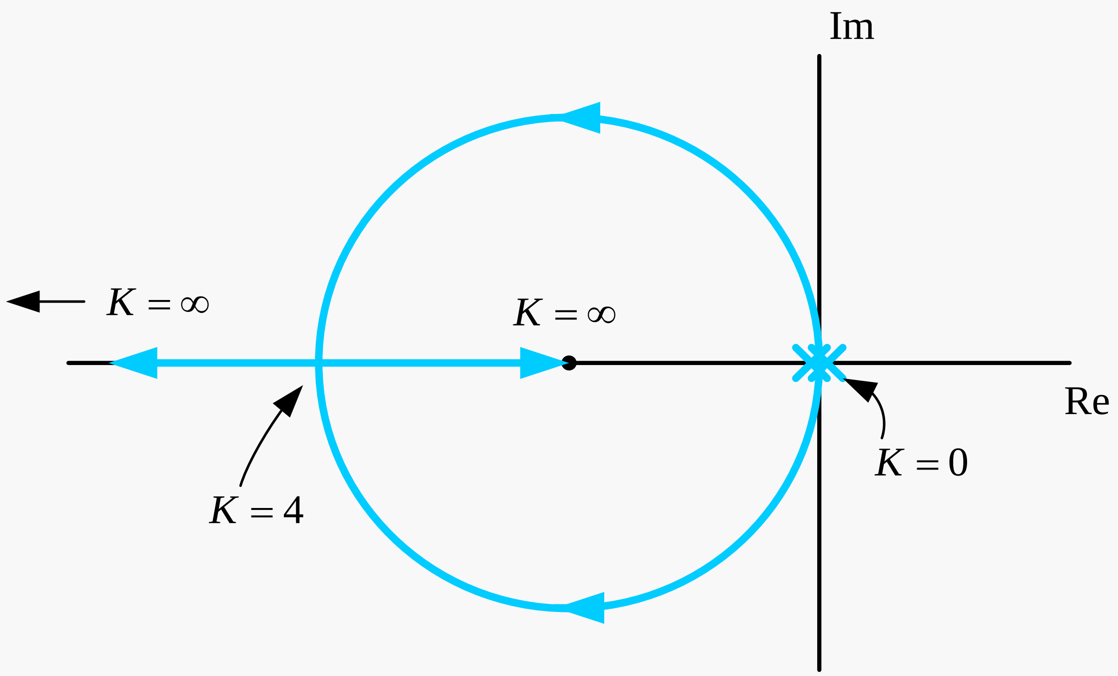


Example

$$G(s) = K\frac{(s+1)}{s}\frac{1}{s}, \quad H(s) = 1 \Rightarrow G_{CL} = \frac{G}{1+G}$$

$$1+G(s) = 1+K\frac{(s+1)}{s}\frac{1}{s} = \frac{s^2+Ks+K}{s^2}$$

$$s^2+Ks+K=0 \Rightarrow s = \frac{1}{2}\left(-K \pm \sqrt{K^2-4K}\right)$$



Transient Response

Consider a system described by the open loop transfer function

$$L(s) = K\frac{n(s)}{d(s)} \xrightarrow{pfd} c_0 + \frac{c_1}{s+\lambda_1} + \cdots +$$

There are three ways to assess system transient behavior:

1. time domain (output time trajectories)
2. pole (or eigenvalue) location
3. frequency response (Bode or Nyquist plots)

The root locus method is concerned with adjusting the closed loop pole positions

Here we consider pole location. The poles are the roots of

$$d(s) = \left(s^2 + 2\rho_1\omega_1 s + \omega_1^2\right)\cdots\left(s^2 + 2\rho_p\omega_p s + \omega_p^2\right)(s+\lambda_1)\cdots(s+\lambda_q) = 0$$

Complex roots occur in complex conjugate pairs. In this case there are 2p+q poles

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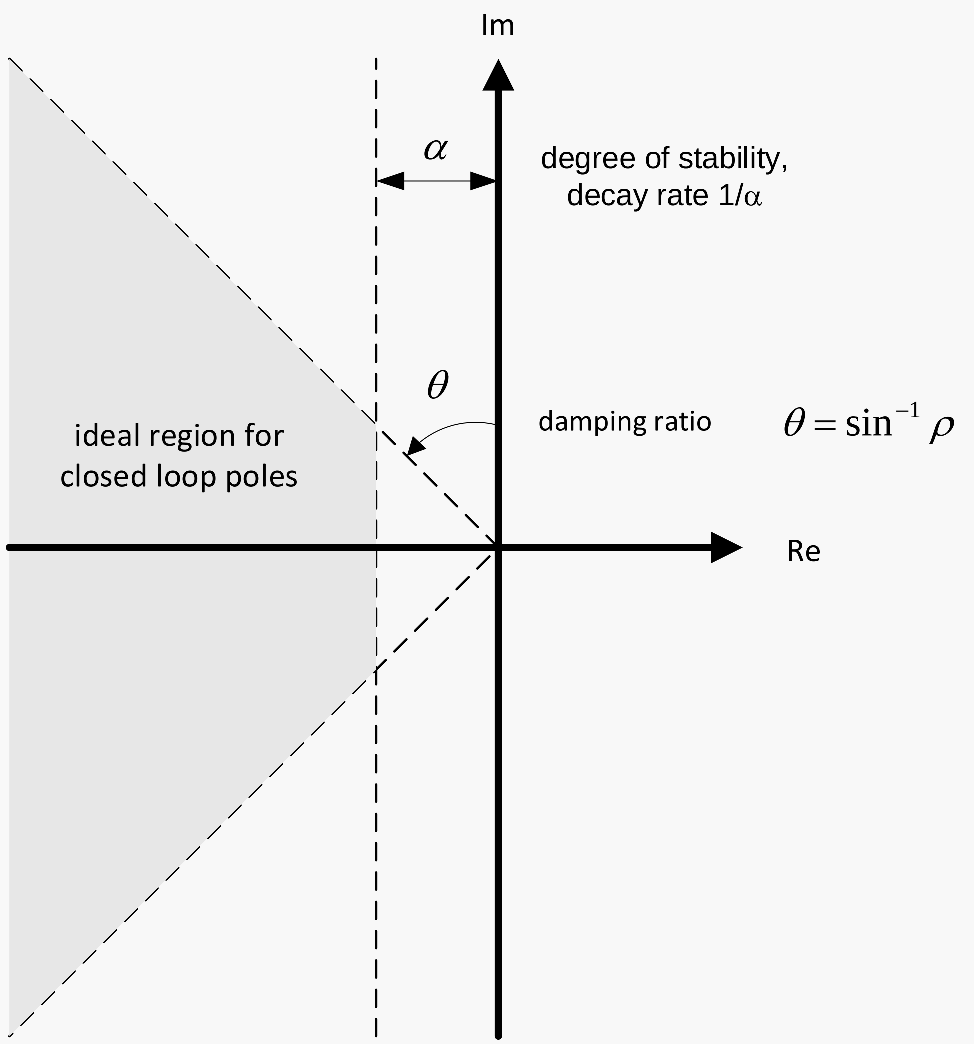
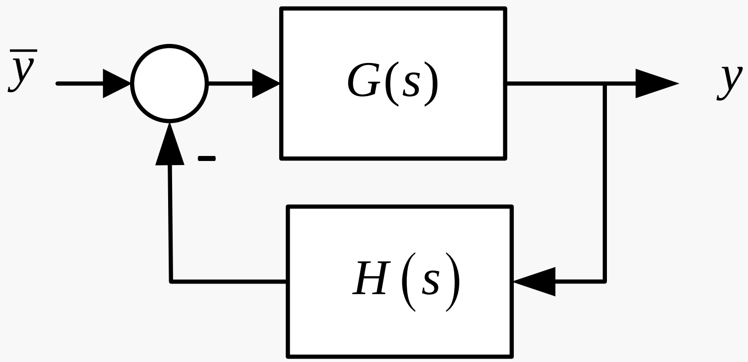
Ideal Pole Locations



Problem Definition

The closed loop input response transfer function is

$$G_{yy}(s)=\frac{G(s)}{1+GH(s)}$$



The error response transfer function is (recall $e=\bar{y}-y$)

$$G_e(s)=1-G_y(s)=\frac{1+GH-G}{1+GH}$$

The poles of the closed loop system are the roots (zeros) of

$$\boxed{1+GH(s)=0}$$

Problem Definition, Cont'd

Suppose

$$G(s)H(s) = K\frac{n(s)}{d(s)} = K\frac{(s-z_1)\cdots(s-z_m)}{(s-p_1)\cdots(s-p_n)} = K\frac{s^m + b_{m-1}s^{m-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$

$n(s), d(s)$ are completely known, but K is a parameter that we can adjust.

Root Locus Problem:

Generate a sketch in the complex plane of the closed loop poles with varying gain K.

Solution Strategies

- We will do this two ways:
  - The easy way: Have MATLAB solve for the roots for each of a specified list of values for K and plot them.
  - The hard (old) way: Generate a sketch by hand.
- Why do it the hard way at all?
  - We need to know how to interpret the plot.
  - We obtain insight concerning the choice of compensator.
  - We learn how to set the compensator parameters other than the gain K.

Root Locus Method ~ 1

$$1+G(s)H(s)=0 \Leftrightarrow \frac{d(s)+Kn(s)}{d(s)}=0 \Rightarrow d(s)+Kn(s)=0$$

$$1+G(s)H(s)=0 \Leftrightarrow K\frac{n(s)}{d(s)}=-1=e^{j(2k+1)\pi},\ k=0,\pm1,\pm2,\ldots$$

This means

Magnitude equation

$$\left|K\frac{n(s)}{d(s)}\right|=1 \text{ and } \angle\left(K\frac{n(s)}{d(s)}\right)=(2k+1)\pi$$

Angle equation

for $K \geq 0$:

$$K\left|\frac{n(s)}{d(s)}\right|=1 \text{ and } \angle\left(\frac{n(s)}{d(s)}\right)=(2k+1)\pi$$

Root Locus Method ~ 2

Our goal is to find values of s that satisfy both of these equations.

$\Rightarrow$ Note that for any given s, the magnitude equation is satisfied for some value of K, i.e.,

$$K = \left| \frac{d(s)}{n(s)} \right|$$

$\Rightarrow$ Note that the angle equation does not depend on K at all.

Strategy:

First, find values of s that satisfy the angle equation.

Second, calibrate the plot using the magnitude equation.

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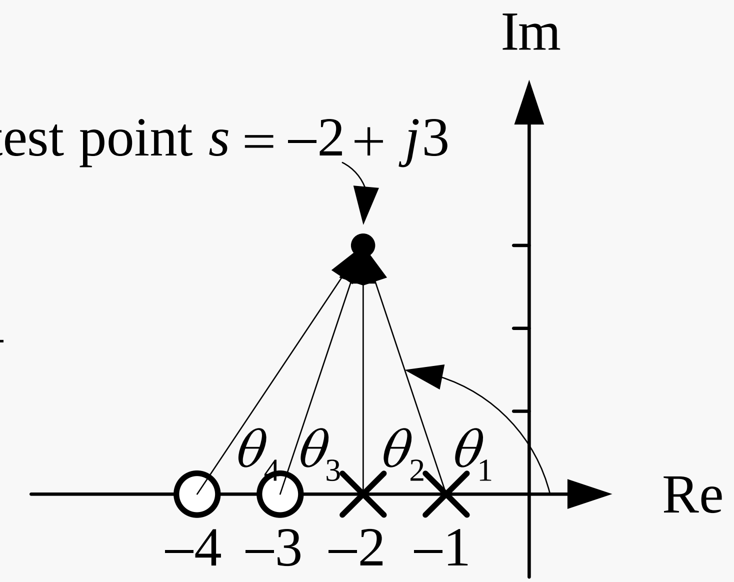
Root Locus ~ 3 Using the Angle Formula

$$G(s)H(s) = \frac{(s+3)(s+4)}{(s+1)(s+2)}$$



$$\angle\left(\frac{n(s)}{d(s)}\right) = (2k+1)\pi$$

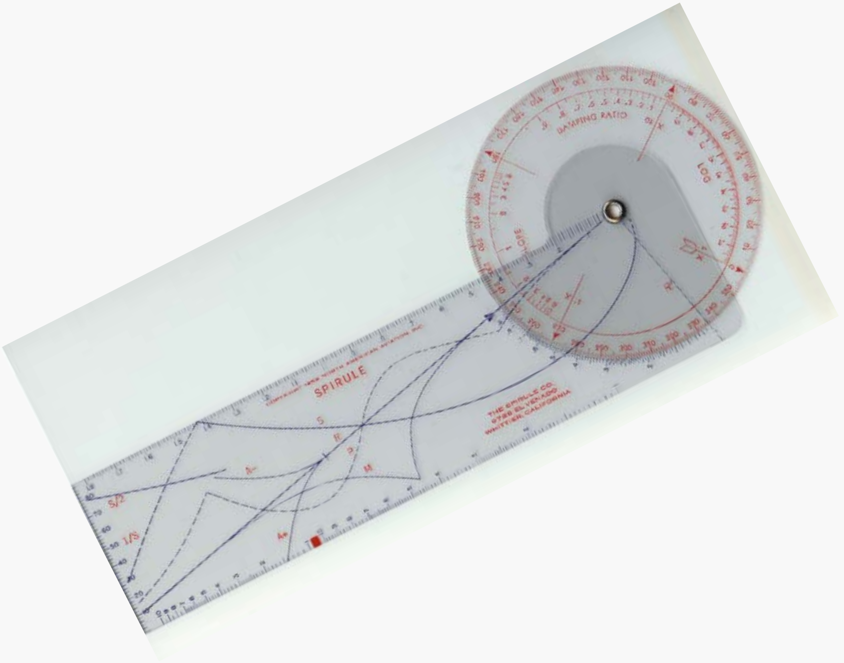
$$\Rightarrow \angle n(s) - \angle d(s) = (2k+1)\pi$$

$$\theta_3 + \theta_4 - (\theta_1 + \theta_2) = 70.55^\circ \neq (2k+1)\pi \text{ for any integer } k$$

$s = -2 + j3$ is not a point on the root locus

The point $s = -2 + i\sqrt{2}/2$ is

$$\theta_3 + \theta_4 - (\theta_1 + \theta_2) = 180^\circ$$



Basic Rules ~ 1

1. Number of branches: The number of branches of the root locus equals the number of open loop poles.

   the order of the polynomial $d(s)+Kn(s)$ is the order of $d(s)$

2. Symmetry: The root locus is symmetric about the real axis.

   Poles occur in complex conjugate pairs.

3. Starting & ending points: The root locus begins at the open loop poles and ends at the finite and infinite open loop zeros.

   $d(s)+Kn(s)=0 \rightarrow d(s)=0$ as $K \rightarrow 0$

   $\frac{1}{K}d(s)+n(s)=0 \rightarrow n(s)=0$ as $K \rightarrow \infty$ if s is bounded

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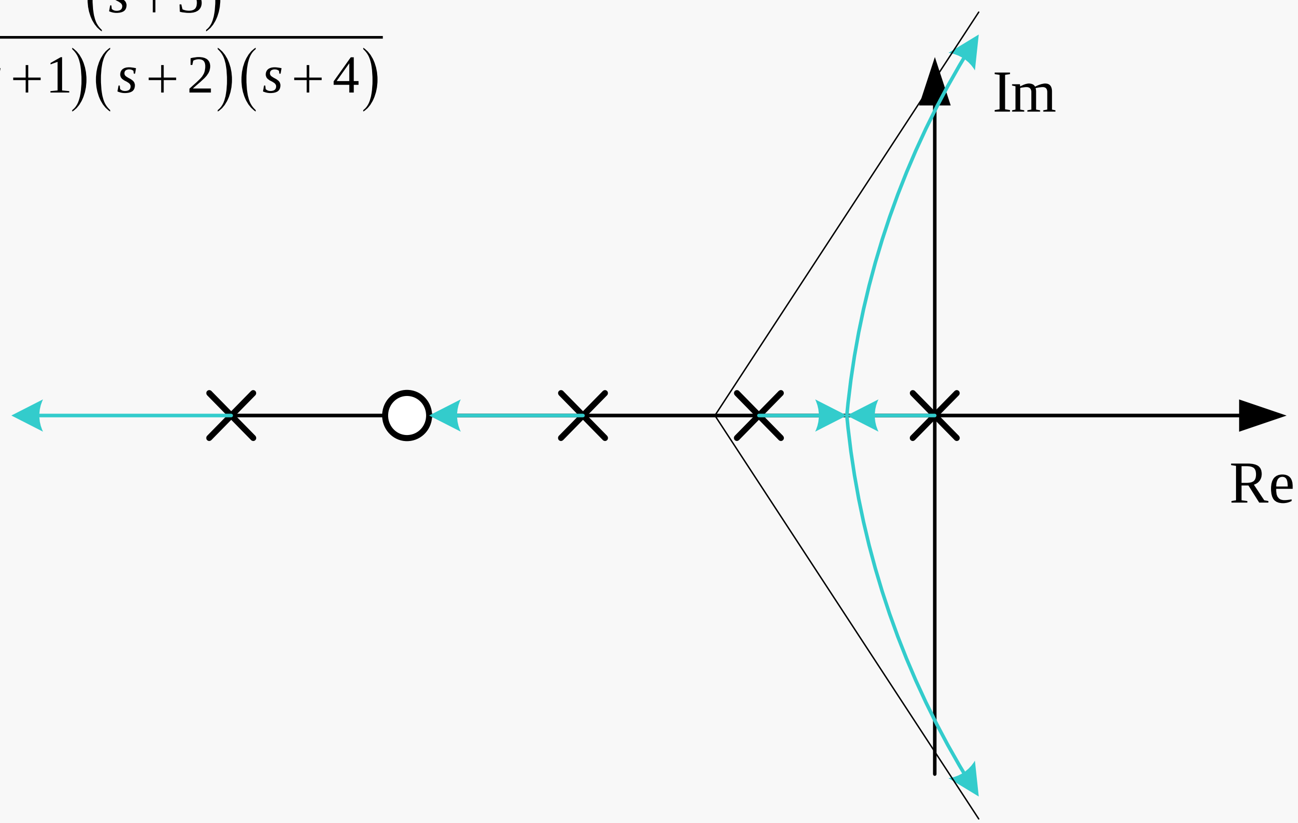
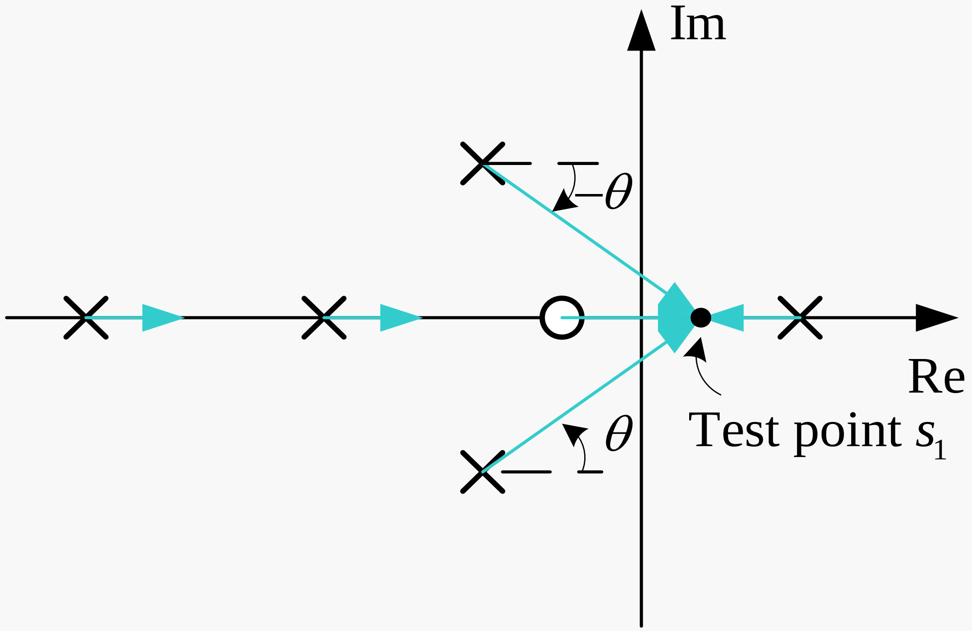
Example:

$$G(s)H(s) = K\frac{(s+3)}{s(s+1)(s+2)(s+4)}$$



Basic Rules ~ 2

4. Real-axis segments: For $K > 0$, real axis segments to the left of an odd number of finite real axis poles and/or zeros are part of the root locus.



Basic Rules ~ 3

5. Behavior at infinity: The root locus approaches infinity along asymptotes with angles:

$$\theta = \frac{(2k+1)\pi}{\#\ finite\ \ poles - \#\ finite\ \ \ zeros},\ k = 0, \pm 1, \pm 2, \pm 3, \ldots$$

Furthermore, these asymptotes intersect the real axis at a common point given by

$$\sigma = \frac{\sum \text{finite}\ \ \text{poles} - \sum \text{finite}\ \ \text{zeros}}{\#\ \ \text{finite}\ \ \text{poles} - \#\ \ \text{finite}\ \ \text{zeros}}$$

Basic Rules ~ 4

Angle part is easy:

$$\angle \frac{n(s)}{d(s)} = \sum_{i=1}^{m} \angle (s - z_i) - \sum_{i=1}^{n} \angle (s - p_i)$$

Take $s = \rho e^{j\theta}$. For $\rho \to \infty$, $\angle (s - \lambda_i) \to \angle s = \theta$

Then $\sum_{i=1}^{m} \angle (s - z_i) - \sum_{i=1}^{n} \angle (s - p_i) \to m\theta - n\theta$

So $\angle \frac{n(s)}{d(s)} = (2k+1)\pi \Rightarrow (m-n)\theta = (2k+1)\pi$

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Basic Rules ~ 4

6. Real axis breakaway and break-in points: The root locus breaks away from the real axis where the gain is a (local) maximum on the real axis, and breaks into the real axis where it is a local minimum. To locate candidate break points

   simply plot $K(s) = \left|\frac{d(s)}{n(s)}\right|$ on the axis segment

   or solve $\frac{d}{ds}\left(\left|\frac{d(s)}{n(s)}\right|\right) = 0$

7. $j\omega$-axis crossings: Use Routh test to determine values of K for which loci cross imaginary axis.

Routh Stability Test

It is desired to determine the number of right hand plane roots of a polynomial, say:

$$s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0$$

$$\begin{array}{c|ccc} s^4 & 1 & a_2 & a_0 \\ s^3 & a_3 & a_1 & 0 \\ s^2 & & & \\ s^1 & & & \\ s^0 & & & \end{array} \Rightarrow \begin{array}{c|ccc} s^4 & 1 & a_2 & a_0 \\ s^3 & a_3 & a_1 & 0 \\ s^2 & b_1 & b_2 & b_3 \\ s^1 & c_1 & c_2 & \\ s^0 & d_1 & & \end{array}$$

$$b_1 = \frac{-\begin{vmatrix} 1 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3}, b_2 = \frac{-\begin{vmatrix} 1 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3}, \ldots$$

$$c_1 = \frac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{a_3}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

The number of right half plane poles is equal to the number of sign changes in the first column.

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Using MATLAB

The basic MATLAB functions are:

rlocus

rlocus(sys) calculates and plots the root locus of the open-loop SISO model sys.

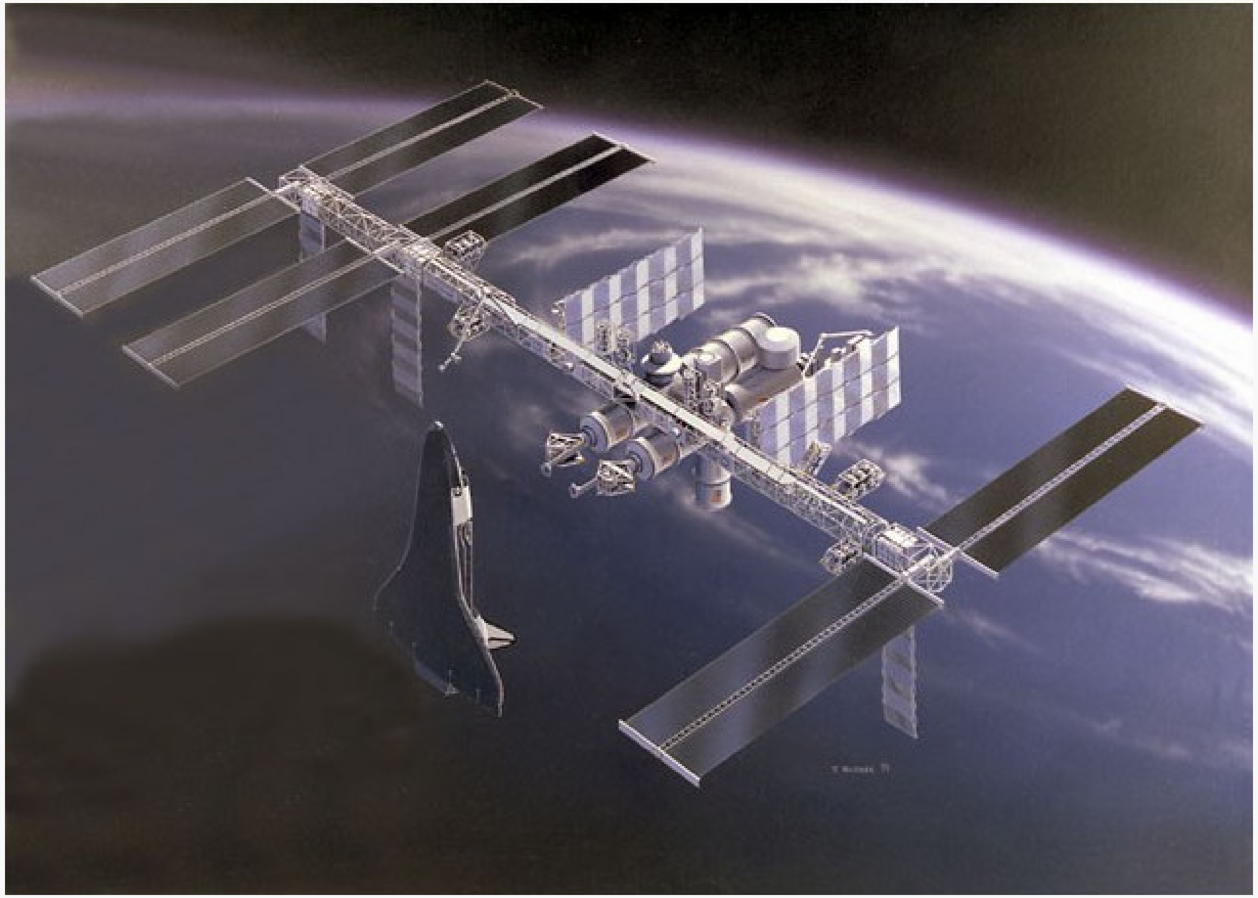
rlocfind

[K,POLES] = RLOCFIND(SYS) is used for interactive gain selection from the root locus plot of the SISO system SYS generated by RLOCUS. RLOCFIND puts up a crosshair cursor in the graphics window which is used to select a pole location on an existing root locus. The root locus gain associated with this point is returned in K and all the system poles for this gain are returned in POLES.

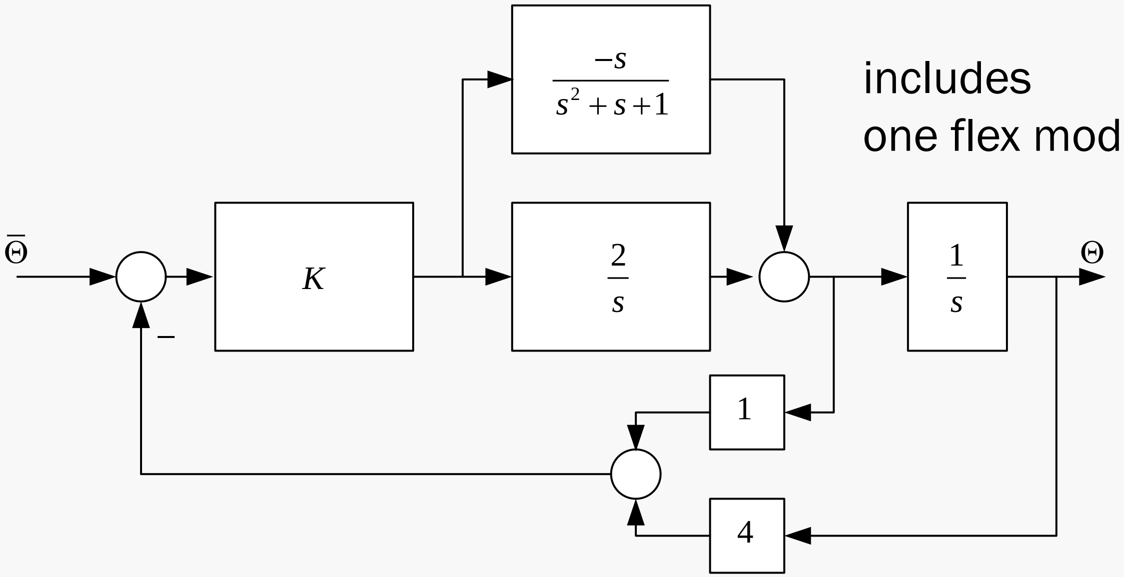
sisotool

When invoked without input arguments, sisotool opens a SISO Design GUI for interactive compensator design. This GUI allows you to design a single-input/single-output (SISO) compensator using root locus and Bode diagram techniques.

Flexible Spacecraft

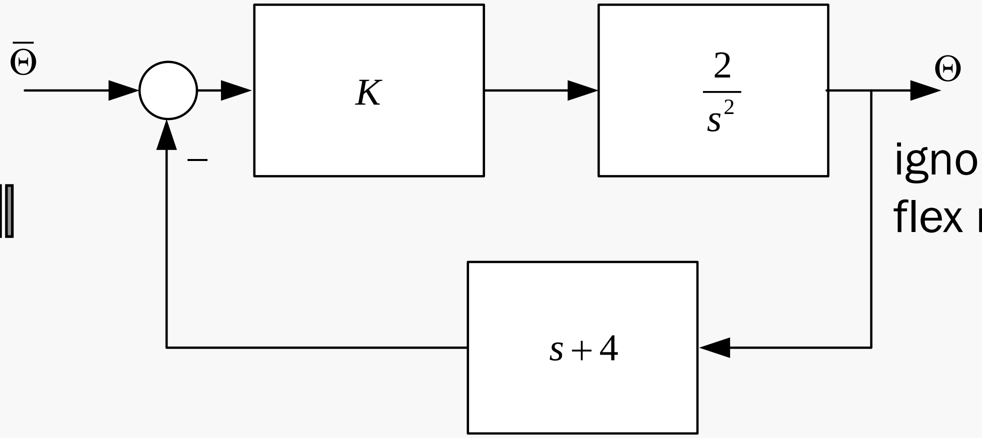


Flexible Spacecraft



$$G(s)H(s) = K\frac{(s+4)(s^2+2s+2)}{s^2(s^2+s+1)}$$

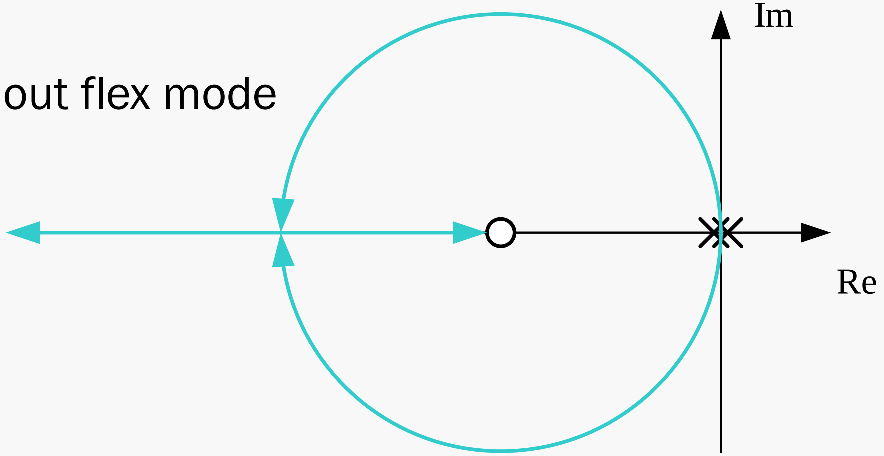
$$= K\frac{(s+4)(s+1\pm j)}{s^2(s+0.5\pm j0.866)}$$



$$G(s)H(s) = K\frac{s+4}{s^2}$$

Spacecraft

without flex mode

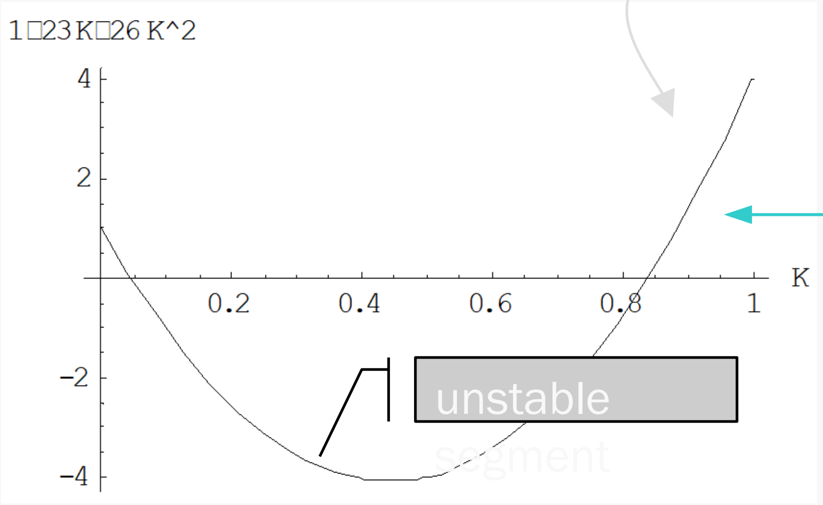


$$d_{cl}(s) = s^4 + (1+K)s^3 + (1+6K)s^2 + 10Ks + 8K$$

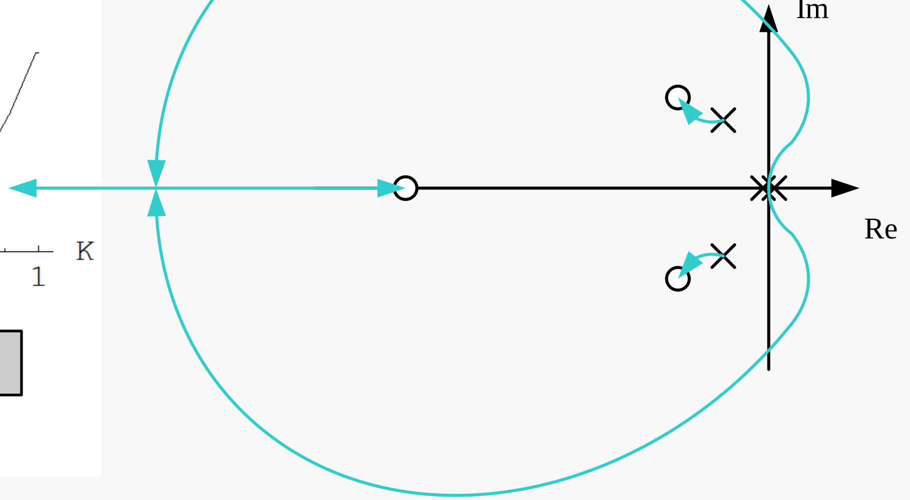
s^4	1	$1+6K$	$8K$
s^3	$1+K$	$10K$	0
s^2	$\dfrac{1-3K+6K^2}{1+K}$	$8K$	0
s^1	$\dfrac{2K\left(1-23K+26K^2\right)}{1-3K+6K^2}$	0	
s^0	$8K$		

Always positive
(prove it)

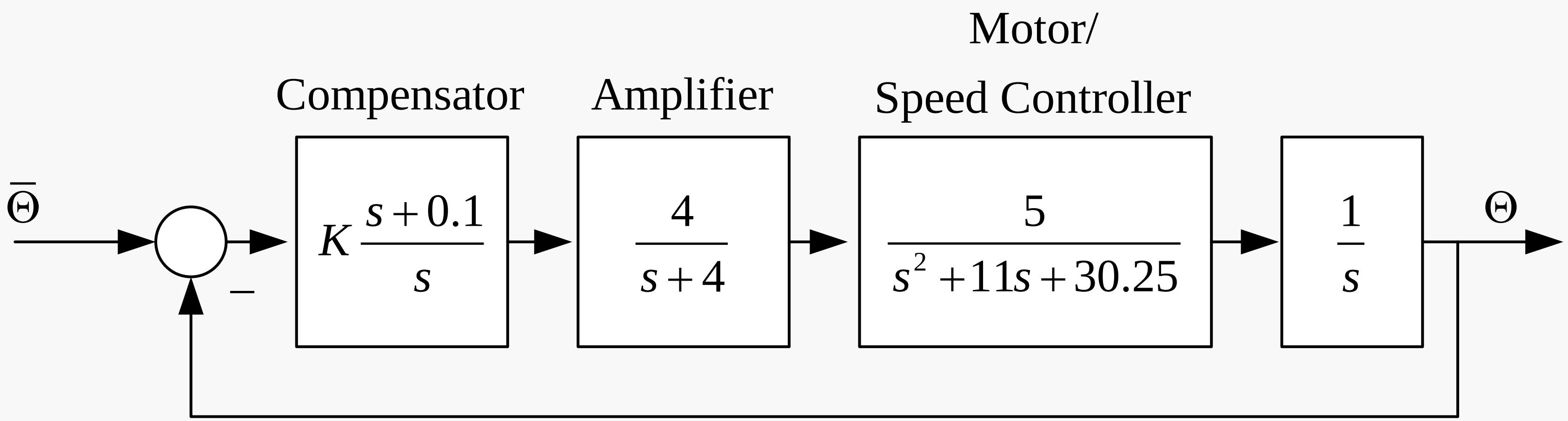
$$1+K \geq 0, \quad 1-3K+6K^2 \geq 0, \quad 1-23K+26K^2 \geq 0$$



with flex mode



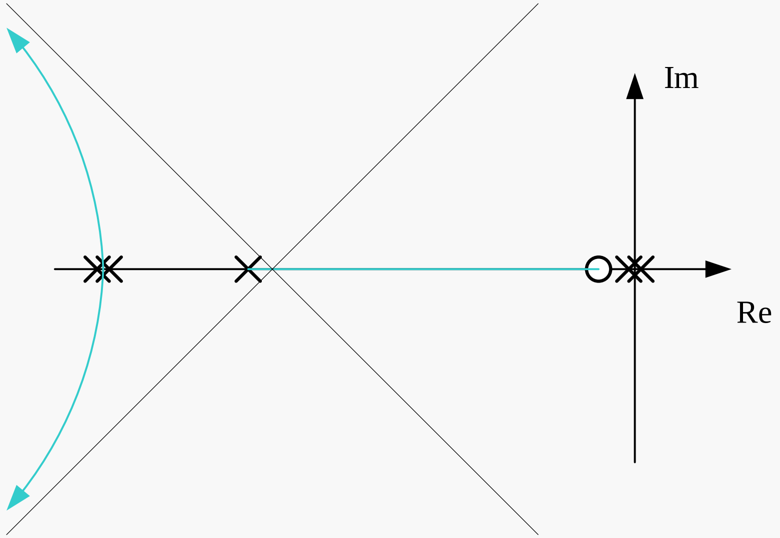
Example Robotic Arm



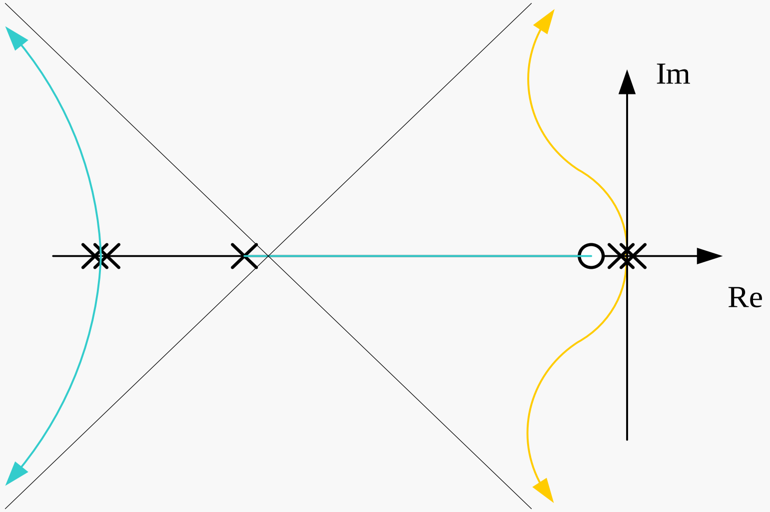
$$L(s) = 20K\frac{s+0.1}{s^2(s+4)(s^2+11s+30.25)} = 20K\frac{s+0.1}{s^2(s+4)(s+5.5)^2}$$

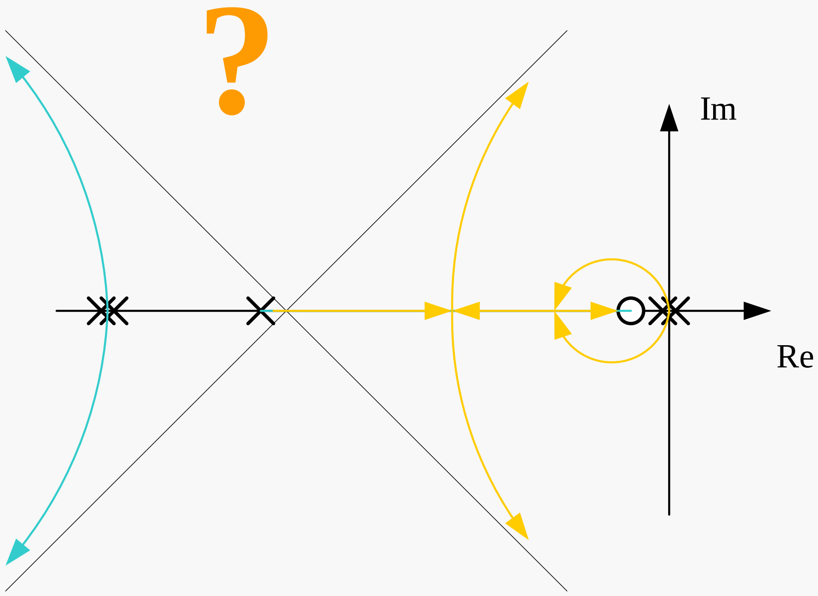
Robotic Arm 2

$$L(s) = 20K\frac{s+0.1}{s^2(s+4)(s+5.5)^2}$$



Asymptotes @ 45, 135, 225, 315
Centroid @ -3.725

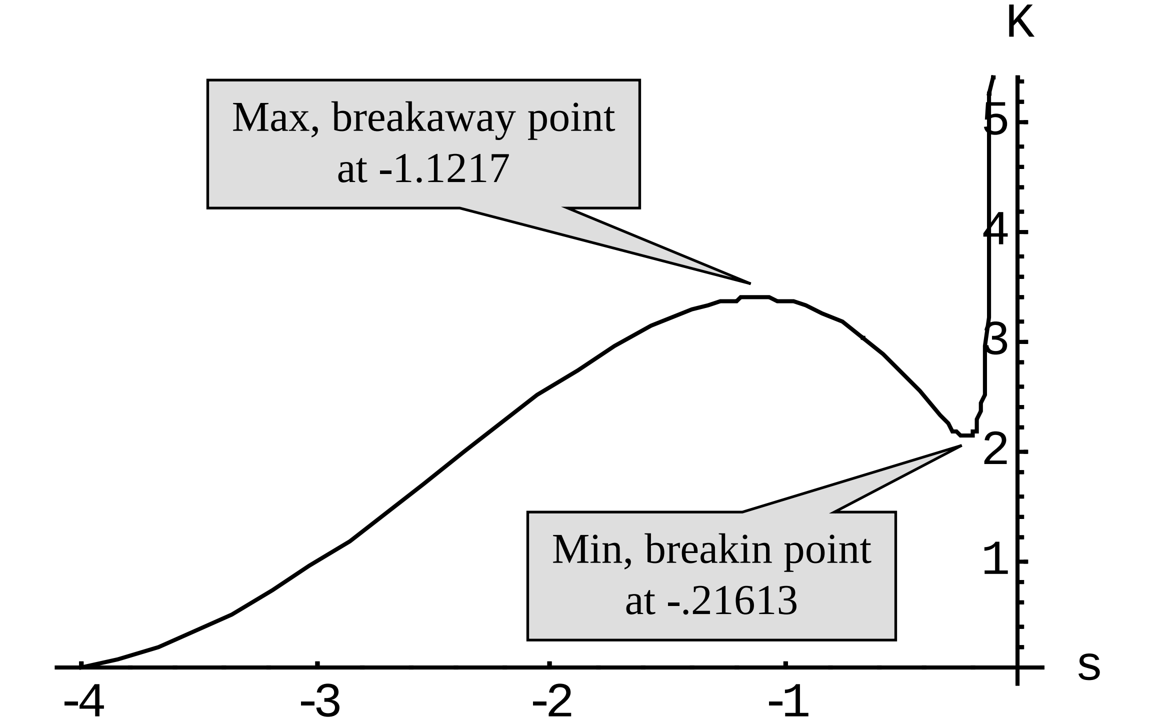


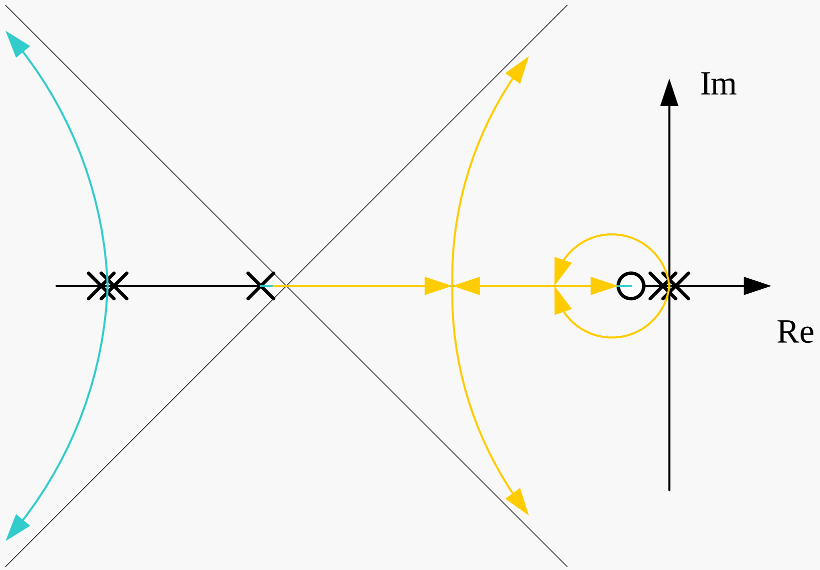


Robotic Arm 3

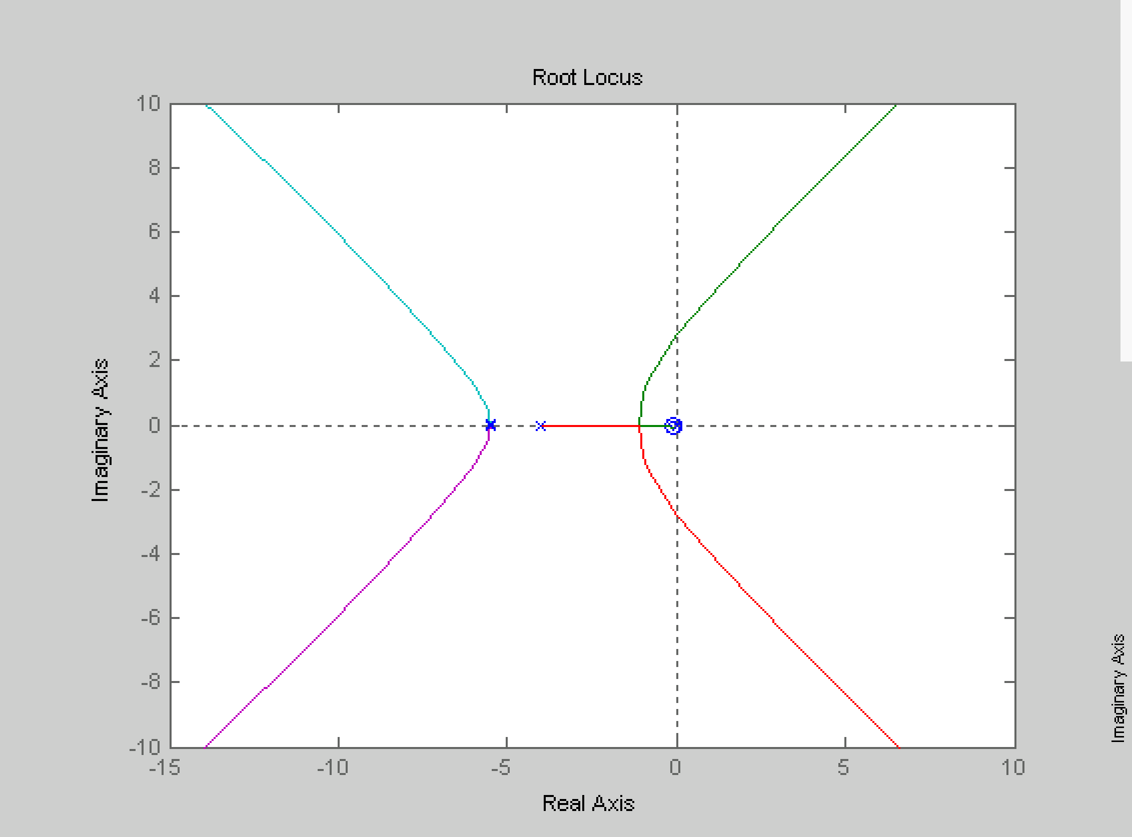
Plot

$$K = \left|\frac{d(s)}{n(s)}\right|, s \in (-4, -0.1)$$

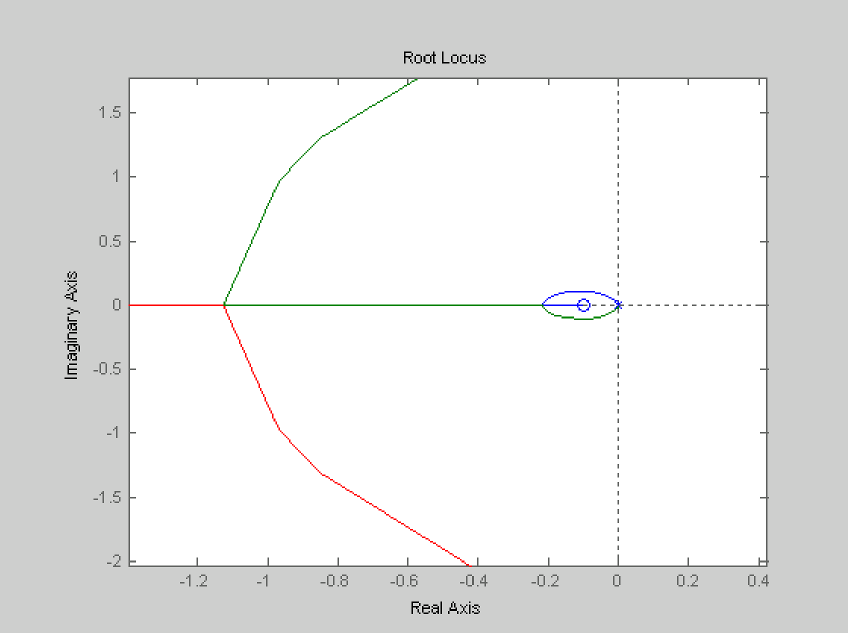




Robotic Arm 4




```
>> s=tf('s');
>> G=20*(s+0.1)/(s^2*(s+4)*(s+5.5)^2);
>> rlocus(G)
>> rlocfind(G)
Select a point in the graphics window
selected_point =
  -0.2161 + 0.0104i
ans =
    2.1211
selected_point =
  -0.0415 + 2.7640i
ans =
   24.9776
```



Robotic Arm 5

$$G_{cl} = \frac{20K(s+0.1)}{s^5 + 15s^4 + 74.25s^3 + 121s^2 + 20Ks + 2K}$$

Routh Table

s^5	1	74	$20K$
s^4	15	121	$2K$
s^3	65.9	$19.86K$	
s^2	$121-4.52K$	$2K$	
s^1	$\dfrac{2271K-89.76K^2}{121-4.52K}$		
s^0	$2K$		

$$121-4.52K>0,\ 2271K-89.76K^2>0,\ K>0$$

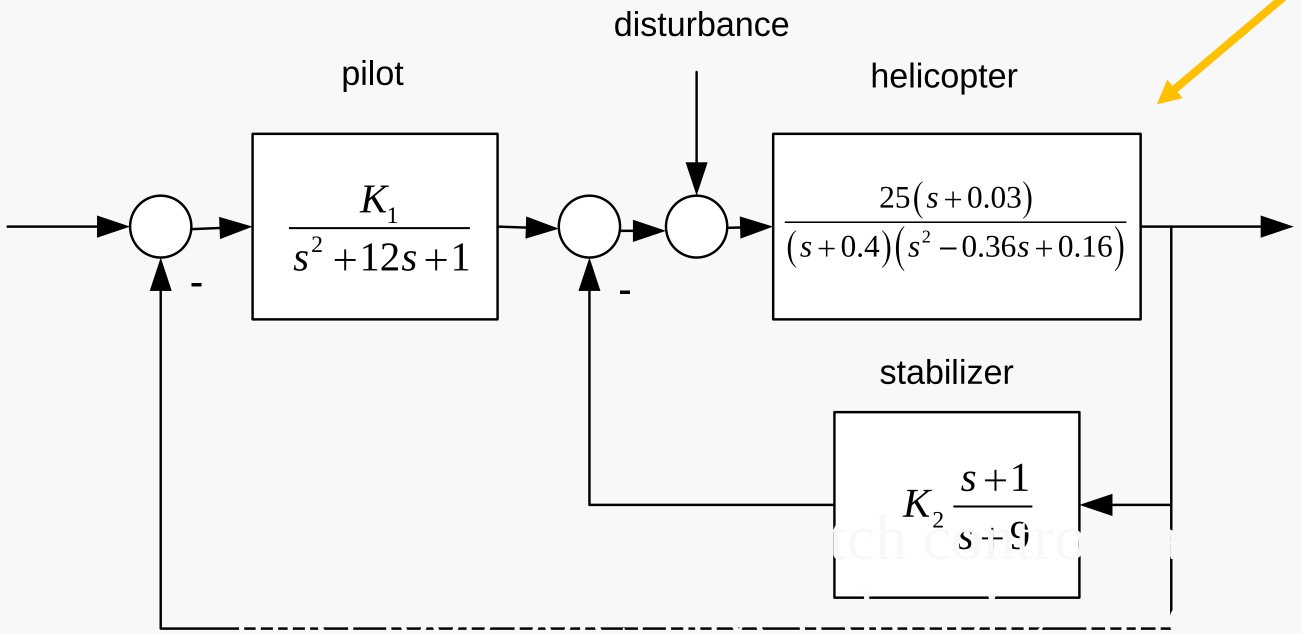
$$K<\frac{121}{4.52}=26.769$$

Governing Inequality

$$\boxed{K<\frac{2271}{89.76}=25.308}$$

Example: Helicopter Pitch Control

Notice unstable dynamics



Inner (Stabilization) Loop

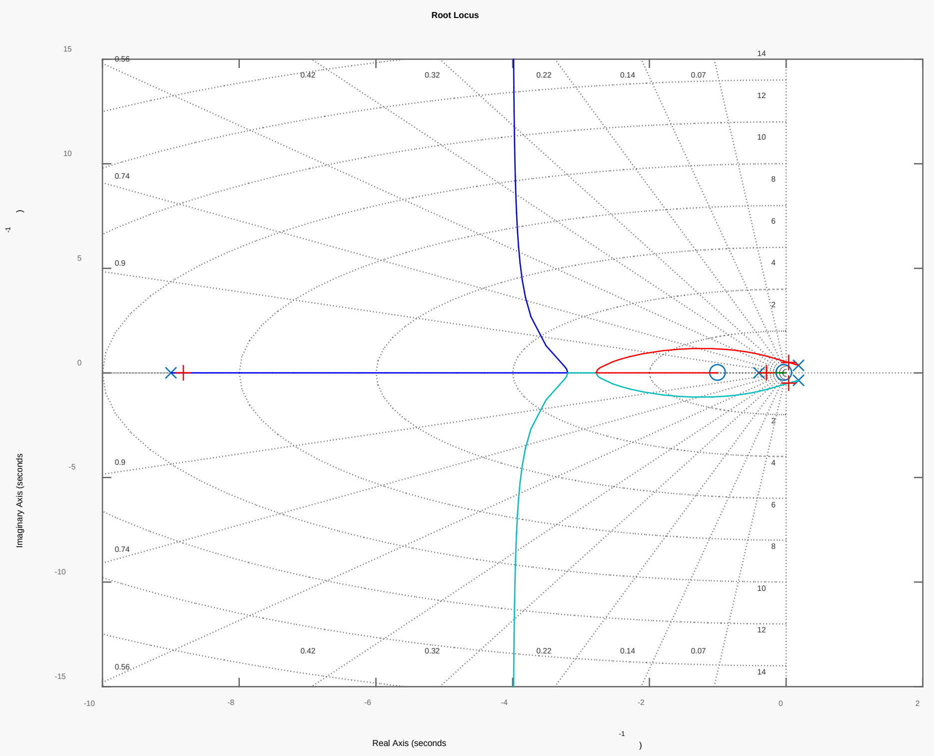
$$1+GH=1+K_2\frac{25(s+0.03)}{(s+0.4)(s^2-0.36s+0.16)}\frac{(s+1)}{(s+9)}$$

The inner loop root locus is shown on the right. Choose a gain $K_2=1.5$.

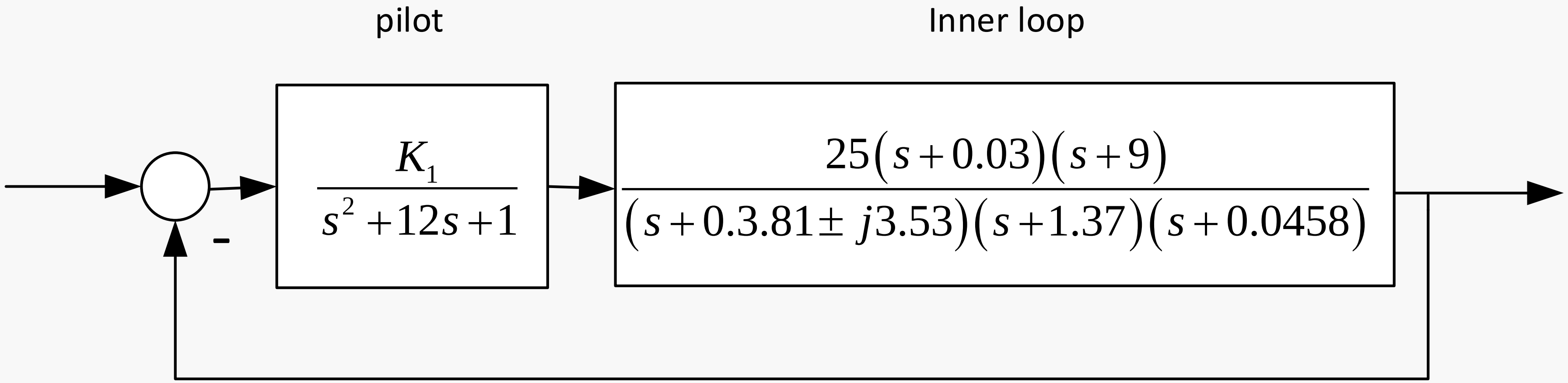
The inner loop resolves to:

$$G_s(\mathrm{s})=\frac{G_p}{1+1.5G_cG_p}$$

$$=\frac{25(s+0.03)(s+9)}{(s+3.81\pm j3.53)(s+1.37)(s+0.0458)}$$



Outer Loop Design

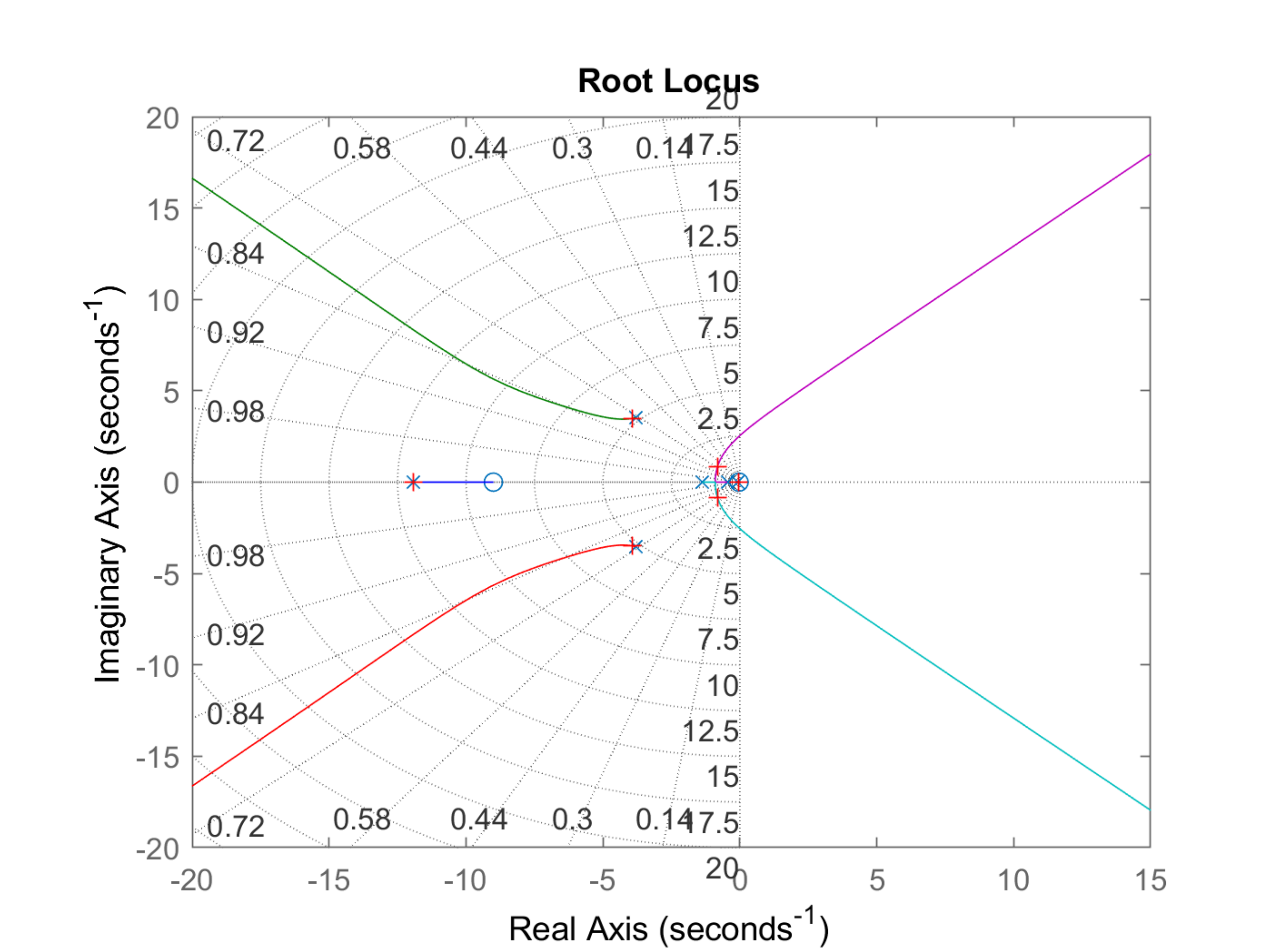


$$L(s)=K_1 25\frac{(s+0.03)(s+9)}{(s^2+12s+1)(s+3.81\pm j3.53)(s+1.37)(s+0.0458)}$$

Outer Loop

- On the basis of the root locus on the right, choose a gain of K=1.
- The closed loop poles are:

$$
\begin{aligned}
&-11.91 \\
&-3.92 \pm j3.50 \\
&-0.819 \pm j0.835 \\
&-0.053
\end{aligned}
$$



Disturbance Response Error

$$G_{de} = \frac{-G_s}{1 + G_{pilot} G_s}$$

$$= \frac{25(s+0.03)(s+9)}{(s+11.91)(s+3.92 \pm j3.50)(s+0.633 \pm j0.646)(s+0.0314)}$$

$$\lim_{s \to 0} s G_{de} \frac{1}{s} = -0.7987$$

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Summary

- How poles characterize transient response
- Observing the influence of gain on closed loop poles using root locus plots
- Sketching the root locus:
  - The magnitude and gain formulas
  - Basic rules of root locus sketching
  - Using MATLAB
- Examples